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Maximal localisation in the presence of minimal uncertainties in positions and in momenta

Haye Hinrichsen*
Department of Physics of Complex Systems
Weizmann Institute
Rehovot 76100, Israel

Achim Kempf†
Department of Applied Mathematics & Theoretical Physics
University of Cambridge
Cambridge CB3 9EW, U.K.

Abstract

Small corrections to the uncertainty relations, with effects in the ultraviolet and/or infrared, have been discussed in the context of string theory and quantum gravity. Such corrections lead to small but finite minimal uncertainties in position and/or momentum measurements. It has been shown that these effects could indeed provide natural cutoffs in quantum field theory. The corresponding underlying quantum theoretical framework includes small 'noncommutative geometric' corrections to the canonical commutation relations. In order to study the full implications on the concept of locality it is crucial to find the physical states of then maximal localisation. These states and their properties have been calculated for the case with minimal uncertainties in positions only. Here we extend this treatment, though still in one dimension, to the general situation with minimal uncertainties both in positions and in momenta.

*supported by Minerva foundation, fehaye@wicc.weizmann.ac.il
†supported by Studienstiftung des Deutschen Volkes, BASF-Fellow, and Research Fellow of Corpus Christi College in the Univ. of Cambridge, a.kempf@amtp.cam.ac.uk

1 Introduction

The short distance structure of conventional geometry can be considered experimentally confirmed up to the order of 1 TeV, see e.g. [1]. In string theory and quantum gravity certain corrections to the short distance structure and the uncertainty relations have been suggested to appear at smaller scales (the latest at the Planck scale), see e.g. [2]-[7] and for a recent review [8].

Here we continue a series of articles [9]-[15] in which are studied the quantum theoretical consequences of small corrections to the canonical commutation relations

$$[\mathbf{x}_i, \mathbf{p}_j] = i\hbar(\delta_{ij} + \alpha_{ijkl}\mathbf{x}_k\mathbf{x}_l + \beta_{ijkl}\mathbf{p}_k\mathbf{p}_l + ...) \tag{1}$$

including the possibility that also $[\mathbf{x}_i, \mathbf{x}_j] \neq 0, \quad [\mathbf{p}_i, \mathbf{p}_j] \neq 0$. A crucial feature of this 'noncommutative geometric' ansatz, which was first studied in [11], is that for appropriate matrices α and β, Eq.1 implies the existence of finite lower bounds to the determination of positions and momenta. These bounds take the form of finite minimal uncertainties Δx_0 and Δp_0, obeyed by all physical states. In fact, the approach covers the case of those corrections to the uncertainty relations which we mentioned above, see [12].

A framework with a finite minimal uncertainty Δx_0 can as well be understood to describe effectively nonpointlike particles, than as describing a fuzzy space. As discussed in [12]-[15] the approach, with appropriately adjusted scales, could have therefore more generally a potential for an effective description of nonpointlike particles, such as e.g. nucleons or quasi-particles in solids.

Analogously, on large scales a minimal uncertainty Δp_0 may offer new possibilities to describe situations where momentum cannot be precisely determined, in particular on curved space [13].

Using the path integral formulation it has been shown in [13] that such noncommutative background geometries can ultraviolet and infrared regularise quantum field theories in arbitrary dimensions through minimal uncertainties $\Delta x_0, \Delta p_0$. However, a complete analysis of the modified short distance structure, and in particular the calculation of the states of maximal localisation, has so far only been carried out for the special case of the commutation relations $[\mathbf{x}, \mathbf{p}] = i\hbar(1 + \beta\mathbf{p}^2)$, in [15]. The reason is that those cases are representation theoretically much easier to handle in which either α or β vanish, i.e. with minimal uncertainties in either positions or in momenta only. We now solve the more general, though still one-dimensional problem, involving both minimal uncertainties in positions and in momenta.

We define the associative Heisenberg algebra $\mathcal{A}$ with corrections parametrised by small constants $\alpha, \beta \geq 0$

$$[\mathbf{x}, \mathbf{p}] = i\hbar(1 + \alpha\mathbf{x}^2 + \beta\mathbf{p}^2) \tag{2}$$

or, in a notation which will prove more convenient ($q \geq 1$)

$$[\mathbf{x}, \mathbf{p}] = i\hbar \left(1 + (q^2 - 1)\left(\frac{\mathbf{x}^2}{4L^2} + \frac{\mathbf{p}^2}{4K^2}\right)\right) \tag{3}$$

where the constants L, K carry units of length and momentum and are related by:

$$4KL = \hbar(1 + q^2) \tag{4}$$

While the first correction term contributes for large $\langle \mathbf{x}^2 \rangle = \langle \mathbf{x} \rangle^2 + (\Delta x)^2$, which is the definition of the infrared, the second correction term contributes for large $\langle \mathbf{p}^2 \rangle = \langle \mathbf{p} \rangle^2 + (\Delta p)^2$, i.e. in the ultraviolet.
The corresponding uncertainty relation

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(1 + \alpha\left((\Delta x)^2 + \langle \mathbf{x} \rangle^2\right) + \beta\left((\Delta p)^2 + \langle \mathbf{p} \rangle^2\right)\right) \tag{5}$$

holds in all $*$-representations of the commutation relations and reveals these infrared and ultraviolet modifications as minimal uncertainties in positions and momenta [12]:

$$(\Delta x_{min})^2 = L^2 \frac{q^2 - 1}{q^2}\left(1 + (q^2 - 1)\left(\frac{\langle \mathbf{x} \rangle^2}{4L^2} + \frac{\langle \mathbf{p} \rangle^2}{4K^2}\right)\right) \tag{6}$$

$$(\Delta p_{min})^2 = K^2 \frac{q^2 - 1}{q^2}\left(1 + (q^2 - 1)\left(\frac{\langle \mathbf{x} \rangle^2}{4L^2} + \frac{\langle \mathbf{p} \rangle^2}{4K^2}\right)\right) \tag{7}$$

In particular, for all physical states i.e. for all $|\psi\rangle \in D$ with $D \subset H$ being any $*$-representation of the commutation relations of $\mathcal{A}$ in a Hilbert space H, there are finite absolutly smallest uncertainties:

$$(\Delta x_{|\psi\rangle}) = \langle \psi | (\mathbf{x} - \langle \mathbf{x} \rangle)^2 | \psi \rangle^{1/2} \geq L\sqrt{1 - q^{-2}} \qquad \forall \quad |\psi\rangle \in D \tag{8}$$

$$(\Delta p_{|\psi\rangle}) = \langle \psi | (\mathbf{p} - \langle \mathbf{p} \rangle)^2 | \psi \rangle^{1/2} \geq K\sqrt{1 - q^{-2}} \qquad \forall \quad |\psi\rangle \in D \tag{9}$$

We will here only deal with the kinematical consequences of possible corrections to the commutation relations. Arbitrary systems can be considered and studies on dynamical systems, including the calculation of the spectra of Hamiltonians and integral kernels such as Green functions have been carried out for example systems in [9, 10]. Compare also with the features of the discretised quantum mechanics studied e.g. in [16, 17, 18]. A very interesting canonical field theoretical approach with a similar motivation is focusing on generalising the uncertainty relations among the coordinates [19].

2 Hilbert space representation

A crucial consequence of Eqs.8,9 is that there are no eigenvectors to $\mathbf{x}$ nor to $\mathbf{p}$ in any space of physical states i.e. in any $*$-representation D of the generalised commutation relations. As is clear from the definition of uncertainties, e.g. $(\Delta x)^2_{|\psi\rangle} = \langle\psi|(\mathbf{x} - \langle\psi|\mathbf{x}|\psi\rangle)^2|\psi\rangle$, eigenvectors to $\mathbf{x}$ or $\mathbf{p}$ could only have vanishing uncertainty in position or momentum. In particular, the commutation relations of A no longer find spectral representations of $\mathbf{x}$ nor of $\mathbf{p}$.

In the situation of $\alpha = 0$ (or $\beta = 0$), i.e. with $\Delta p_0 = 0$ (or $\Delta x_0 = 0$) there is still the momentum (or position) representation of $\mathcal{A}$ available, in which case the maximal localisation states have been calculated in [15]. Let us now perform the analogous studies for the general case with $\alpha, \beta > 0$, where position and momentum space representations are both ruled out.

To this end we use a Hilbert space representation of $\mathcal{A}$ on a generalised Fock space. The position and momentum operators can be represented as

$$\mathbf{x} = L(a^\dagger + a) \qquad\qquad \mathbf{p} = iK(a^\dagger - a) \tag{10}$$

where the a and $a^\dagger$ obey generalised commutation relations

$$aa^\dagger - q^2 a^\dagger a = 1 \tag{11}$$

and act on the domain D of physical states $D := \{|\psi\rangle = \text{polynomial}(a^\dagger)|0\rangle\}$ as:

$$\begin{aligned} a\,|0\rangle &= 0 \\ a^\dagger|n\rangle &= \sqrt{[n+1]}\,|n+1\rangle \\ a|n\rangle &= \sqrt{[n]}\,|n-1\rangle \end{aligned} \tag{12}$$

where $[n]$ denotes the partial geometric sum or 'q'- number

$$[n] = \frac{q^{2n} - 1}{q^2 - 1} \tag{13}$$

and where the $|n\rangle := ([n]!)^{-1/2}(a^\dagger)^n|0\rangle,\ n = 1, ..., \infty$ are orthonormalised

$$\langle n_1|n_2\rangle = \delta_{n_1,n_2} \tag{14}$$

and D is analytic and dense in the Hilbert space $H = l^2$.

While $\mathbf{x}$ and $\mathbf{p}$ ordinarily are essentially self-adjoint, they are now merely symmetric, which is sufficient to insure that all expectation values are real. The deficiency indices of $\mathbf{x}$ and $\mathbf{p}$ are $(1,1)$, implying the existence of 1-parameter families of self-adjoint extensions. While ordinarily self-adjoint extensions, e.g. for a particle in a box, need to and can be fixed, there is now the subtle effect of the self-adjoint extensions

not being on common domains, which prevents the diagonalisation of $\mathbf{x}$ or $\mathbf{p}$ on physical states, as can also be understood through the uncertainty relations. For the full functional analytical details see [12], where these structures have first been found. We will come back to these functional analytical studies in Sec.6 where we will explicitly calculate the diagonalisations in H. They are of use for the calculation of inverses of $\mathbf{x}$ and $\mathbf{p}$, which are not only needed to describe certain quantum mechanical potentials, but ultimately also to invert kinetic terms e.g. of the form $\mathbf{p}^2 - m^2$ to obtain propagators from the field theoretical path integral, see [13].

3 Maximal localisation states

The absence of eigenvectors of $\mathbf{x}$ or $\mathbf{p}$ in all $*$-representations D of the commutation relations physically implies the absence of absolute localisability in position or momentum i.e. there are no physical states that would have $\Delta x = 0$ or $\Delta p = 0$. More precisely, the uncertainty relation, holding in all D, implies a 'minimal uncertainty gap':

$$\nexists\, |\psi\rangle \in D:\ \Delta x_{|\psi\rangle} < \Delta x_0 \qquad \text{and} \qquad \nexists\, |\psi\rangle \in D:\ \Delta p_{|\psi\rangle} < \Delta p_0\ . \tag{15}$$

The state of maximal localisation in position $|\psi_x^{ml}\rangle$ with given position expectation x and vanishing momentum expectation, is defined through

$$\langle\psi_x^{ml}|\mathbf{x}|\psi_x^{ml}\rangle = x\ , \qquad \langle\psi_x^{ml}|\mathbf{p}|\psi_x^{ml}\rangle = 0\ , \qquad (\Delta x)_{|\psi_x^{ml}\rangle} = \Delta x_{min}\ . \tag{16}$$

Explicitly the minimal uncertainty in position then reads

$$(\Delta x)^2_{|\psi_x^{ml}\rangle} = L^2\, \frac{q^2-1}{q^2} \left(1 + (q^2-1)\frac{\langle\mathbf{x}\rangle^2}{4L^2}\right) \tag{17}$$

with the corresponding (now not infinite) uncertainty in momentum:

$$(\Delta p)^2_{|\psi_x^{ml}\rangle} = K^2\, \frac{(q^2+1)^2}{q^2(q^2-1)} \left(1 + (q^2-1)\frac{\langle\mathbf{x}\rangle^2}{4L^2}\right)\ . \tag{18}$$

We focus on maximal localisation in x, the case of maximal localisation in p is fully analogous.

As shown in [15] a state of maximal localisation is determined by the equation

$$\Big((\mathbf{x} - \langle\mathbf{x}\rangle) + i\alpha(\mathbf{p} - \langle\mathbf{p}\rangle)\Big)\,|\psi_x^{ml}\rangle \ = \ 0 \tag{19}$$

where $\alpha = \Delta x / \Delta p$. Inserting Eqs.17,18 we obtain

$$\alpha \ = \ \frac{L(q^2-1)}{K(q^2+1)} \tag{20}$$

so that the condition reads:

$$\left(\frac{q^2+1}{L}\,(\mathbf{x} - \langle\mathbf{x}\rangle) \ + \ i\frac{q^2-1}{K}\,\mathbf{p}\right)|\psi_x^{ml}\rangle \ = \ 0\ . \tag{21}$$

3.1 Maximal localisation states in the Fock basis

In order to explicitly calculate those states that realise the now maximally possible localisation we expand the $|\psi_x^{ml}\rangle$ in the Fock basis

$$|\psi_x^{ml}\rangle \ := \ \frac{1}{\mathcal{N}(x)} \sum_{n=0}^{\infty} q^{-3n/2}\, c_n(x)\, |n\rangle \tag{22}$$

where the $c_n(x)$ are real coefficients and

$$\mathcal{N}(x) \ := \ \sum_{n=0}^{\infty} q^{-3n}\, c_n^2(x) \tag{23}$$

is a normalisation factor (the inserted factors $q^{-3n/2}$ will be convenient later).
The condition for maximal localisation Eq.21 reads in the Fock representation:

$$\Big((q^2+1)(a^\dagger + a - \frac{x}{L}) \ - \ (q^2-1)(a^\dagger - a)\Big)\, |\psi_x^{ml}\rangle \ = \ 0 \ . \tag{24}$$

Inserting the ansatz Eq.22 we are led to the recursion relation

$$\frac{q+q^{-1}}{2L}\, x\, c_n(x) \ = \ \sqrt{q^{-1}[n+1]}\, c_{n+1}(x) \ + \ \sqrt{q[n]}\, c_{n-1}(x) \ . \tag{25}$$

Together with

$$c_{-1}(x) = 0 \qquad \text{and} \qquad c_0(x) = 1 \tag{26}$$

the coefficients $c_n(x)$ are therefore determined as polynomials of degree n in x.

3.2 Relation to continuous q-Hermite polynomials

The coefficients $c_n(x)$ are related to the so-called continuous q-Hermite polynomials. An excellent review on these and other q-orthogonal polynomials is [20].
We use the notation of shifted q-factorials [20]

$$(a;q^2)_n \ := \ \prod_{k=0}^{n-1} (1 - aq^{2k}) \tag{27}$$

which obey the identity

$$(a;q^2)_n \ = \ (-a)^n\, q^{n(n-1)}\, (a^{-1};q^{-2})_n \, . \tag{28}$$

Furthermore we define for later convenience

$$j(x) := \frac{\operatorname{arcsinh}(\omega x)}{\ln q} \ , \qquad\qquad x(j) = \frac{q^j - q^{-j}}{2\omega} \tag{29}$$

where

$$\omega \;:=\; \frac{1}{4L}\,(q+q^{-1})\sqrt{q^2-1}\,. \tag{30}$$

The continuous q-Hermite polynomials $H_n(z|q^2)$ are defined through

$$H_{-1}(z|q^2)=0, \qquad H_0(z|q^2)=1 \tag{31}$$

and the recurrence relation, see Ref.[20]:

$$2zH_n(z|q^2) \;=\; H_{n+1}(z|q^2) \,+\, (1-q^{2n})H_{n-1}(z|q^2)\;. \tag{32}$$

It is not difficult to check that this recursion relation can be brought into the form of the recursion relation Eq.25 for the coefficients $c_n(x)$, by expressing them in terms of the $H_n(z|q^2)$ as:

$$c_n(x) \;=\; \sqrt{\frac{q^n}{[n]!\,(q^2-1)^n}}\; i^{-n} H_n\Big(i\omega x\,|\,q^2\Big)\;. \tag{33}$$

As shown in [20], the continuous q-Hermite polynomials $H_n(z|q^2)$ can be written as

$$H_n(z|q^2) \;=\; \sum_{k=0}^{n}\binom{n}{k}_{q^2} e^{i(n-2k)\theta}\,, \qquad\qquad z=\cos\theta \tag{34}$$

with the q-binomial coefficients:

$$\binom{n}{k}_{q^2} \;=\; \frac{(q^2;q^2)_n}{(q^2;q^2)_k\,(q^2;q^2)_{n-k}}\;. \tag{35}$$

Inserting Eq.34 into Eq.33 and replacing $[n]!$ by

$$[n]! \;=\; \frac{(-)^n(q^2;q^2)_n}{(q^2-1)^n} \;=\; \frac{q^{n^2}(q^{-2};q^{-2})_n}{(q-q^{-1})^n} \tag{36}$$

yields

$$c_n(x) \;=\; \frac{1}{\sqrt{q^{n^2}(q^{-2};q^{-2})_n}}\, i^{-n}\sum_{k=0}^{n}\binom{n}{k}_{q^2} e^{i(n-2k)\theta}\,, \qquad\qquad i\omega x=\cos\theta\;. \tag{37}$$

Because of $i\omega x = \frac{1}{2}(q^{j(x)}-q^{-j(x)})$ we may also write $e^{i\theta}=i\,q^{j(x)}$ and therefore obtain the following exact expression for the coefficients $c_n(x)$:

$$c_n(x) \;=\; \frac{1}{\sqrt{q^{n^2}\,(q^{-2};q^{-2})_n}}\sum_{k=0}^{n}\binom{n}{k}_{q^2}(-)^k\,q^{(n-2k)\,j(x)}\;. \tag{38}$$

We derive further useful properties of the $c_n(x)$.

Classical limit

For $q \to 1$ the recursion relation Eq.25 reduces to

$$\frac{x}{L} c_n(x) \;=\; \sqrt{n+1}\, c_{n+1}(x) \;+\; \sqrt{n}\, c_{n-1}(x)\;. \tag{39}$$

By substituting $x = L\sqrt{2}\, z$ and $H_n(z) = \sqrt{n!\, 2^n}\, c_n(x)$ we obtain $H_0(z) = 1$ and

$$2z\, H_n(z) \;=\; H_{n+1}(z) + 2n\, H_{n-1}(z) \tag{40}$$

which is the defining recursion relation for classical Hermite polynomials $H_n(z)$. Thus the classical limit of the polynomials $c_n(x)$ is given by

$$\lim_{q\to 1} c_n(x) \;=\; \frac{1}{\sqrt{n!\, 2^n}}\, H_n\Big(\frac{x}{L\sqrt{2}}\Big)\;. \tag{41}$$

Representation by the formula of Rodriguez

As a short notation we write $x(j)$ as x_j. Then, introducing the q-difference operator

$$D\, f(x_j) \;=\; \frac{f(x_{j+1}) - f(x_{j-1})}{x_{j+1} - x_{j-1}} \tag{42}$$

the polynomials $c_n(x_j)$ can be expressed as

$$c_n(x_j) \;=\; \frac{(-)^n}{\kappa_n}\, q^{j^2}\, D^n\, q^{-j^2} \tag{43}$$

where

$$\kappa_n \;=\; \sqrt{q^{-n^2}\, [n]!}\, \Big(\frac{q+q^{-1}}{2L}\Big)^n\;. \tag{44}$$

Eq.43 generalises the formula of Rodriguez $H_n(x) = (-)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$ for classical Hermite polynomials. A proof for Eq.43 is outlined in appendix A.

q-difference equation

The generalised formula of Rodriguez Eq.43 implies that

$$c_n(x_{j+1}) - c_n(x_{j-1}) \;=\; \sqrt{q(1-q^{-2n})}\, (q^j + q^{-j})\, c_{n-1}(x_j) \tag{45}$$

which generalises the differentiation rule $\frac{d}{dx} H_n(x) = n H_{n-1}(x)$ for classical Hermite polynomials. In order to prove this equation, we rewrite its l.h.s. using Eqs.42,43

$$c_n(x_{j+1}) - c_n(x_{j-1}) \;=\; (x_{j+1} - x_{j-1})\, D\, c_n(x_j) \;=\; (x_{j+1} - x_{j-1})\, D\, \frac{(-)^n}{\kappa_n}\, q^{j^2}\, D^n\, q^{-j^2}\;. \tag{46}$$

Carrying out the first differentiation on the r.h.s. of this formula (c.f. Eq.91), one obtains a linear combination of the polynomials $c_n(x_j)$ and $c_{n+1}(x_j)$ which in turn can be expressed through the recurrence relation Eq.25 in terms of $c_{n-1}(x_j)$.

It can also be shown by induction that Eq.45 implies the following q-difference equation for the polynomials $c_n(x)$

$$q^j c_n(x_{j-1}) + q^{-j} c_n(x_{j+1}) \;=\; q^{-n}\,(q^j + q^{-j})\,c_n(x_j) \tag{47}$$

which corresponds to the differential equation $2xH_n'(x) - H_n''(x) = 2nH_n(x)$ for classical Hermite polynomials.

Orthogonality

The polynomials $c_n(x)$ obey the orthogonality relation

$$\sum_{j=-\infty}^{\infty} (x_{2j+\kappa+1} - x_{2j+\kappa-1})\, q^{-(2j+\kappa)^2}\, c_m(x_{2j+\kappa}) c_n(x_{2j+\kappa}) \;=\; N_\kappa\, q^n\, \delta_{m,n} \tag{48}$$

where

$$N_\kappa \;=\; \sum_{j=-\infty}^{\infty} (x_{2j+\kappa+1} - x_{2j+\kappa-1})\, q^{-(2j+\kappa)^2}\;. \tag{49}$$

The parameter $0 \le \kappa \le 2$ can be choosen arbitrarily and fixes a family of positions occurring in the sum.
Eq.48 can be proved as follows. The case $m = n = 0$ is trivial. For $n = 0$ and $m > 0$ one can show that the l.h.s. of Eq.48 is equal to

$$\sum_{j=-\infty}^{\infty} (x_{2j+\kappa+1} - x_{2j+\kappa-1})\, D^m\, q^{-(2j+\kappa)^2} \;=\; D^{m-1}\, q^{-j^2}\Big|_{j=-\infty}^{j=+\infty} \;=\; 0\;. \tag{50}$$

Keeping m fixed, a further induction for $n > 0$ completes the proof.

Generating function

A generating function of the polynomials $c_n(x)$ is

$$(t\,q^{-j(x)}; q^{-2})_\infty(-t\,q^{j(x)}; q^{-2})_\infty \;=\; \sum_{n=0}^{\infty} \frac{c_n(x)}{\sqrt{q^{n(n-2)}\,(q^{-2}; q^{-2})_n}}\, t^n\;. \tag{51}$$

In order to verify this expression, we use the q-difference equation Eq.47 and get

$$\begin{aligned} &q^j\,(t\,q^{-j+1}; q^{-2})_\infty\,(-t\,q^{j-1}; q^{-2})_\infty \;+\; q^{-j}\,(t\,q^{-j-1}; q^{-2})_\infty\,(-t\,q^{j+1}; q^{-2})_\infty \\ &=\; (q^j + q^{-j})\,(t\,q^{-j-1}; q^{-2})_\infty\,(-t\,q^{j-1}; q^{-2})_\infty \end{aligned} \tag{52}$$

which in turn can be proved by inserting the definition of the q-factorials (c.f. Eq.27).

4 Quasi- position and momentum wave functions

Generally, all information on position and momentum is contained in the matrix elements of the position and momentum operators, and matrix elements can of course be calculated in arbitrary bases, such as also the Fock basis. Ordinarily, the position and the momentum information content of a state $|\phi\rangle$ of the particle is easily obtained by writing the state as a position or momentum space wave function $\phi(x) = \langle x|\phi\rangle$ or $\phi(p) = \langle p|\phi\rangle$, which is to project onto position or momentum eigenstates, i.e. to project onto states of maximal localisation in x or p.

In the new setting we can now project arbitrary states $|\phi\rangle$ onto the states which realise the maximally possible localisation in position (or in momentum), which are given by Eqs.16,22,38. We call the collection of these projections the quasi-position wavefunction $\phi(x)$ of $|\phi\rangle$:

$$\phi(x) := \langle\psi_x^{ml}|\phi\rangle \tag{53}$$

Here $\phi(x)$ yields the probability amplitude for finding the particle in a state of maximal localisation around the position x with vanishing momentum expectation. As is easily seen from Eqs.16, 21,24 the generalisation to arbitrary momentum expectations is straightforward. The framework for quasi-momentum wave functions

$$\phi(p) := \langle\psi_p^{ml}|\phi\rangle \tag{54}$$

is analogous with $\phi(p)$ being the probability amplitude for finding the particle in a state of maximal localisation in its momentum, with the momentum expectation p and vanishing position expectation (again the definition may easily be generalised to include arbitrary position expectations).

Aiming at the calculation of examples of quasi-wave functions, we need to complete our studies on the maximal localisation states by calculating their norm and scalar product. To this end an important technical tool will be the Christoffel Darboux theorem, for the application of which we will need the limiting cases of the coefficients $c_n(x)$ of the maximal localisation states.

4.1 Limits of $(-1)^n c_{2n}(x)$ and $(-1)^n c_{2n+1}(x)$ for $n \to \infty$

As we prove in appendix B, the polynomials $c_n(x)$, for all odd and for all even n have the nontrivial property that their limit for $n \to \infty$ exists. Denoting again $x_j := x(j)$ these limits are:

$$c^+(x_j) = \lim_{m\to\infty} (-)^m\, c_{2m}(x_j) = A\, q^{j^2/2}\, \theta_2(\frac{\pi j}{2}, \lambda) \tag{55}$$

$$c^-(x_j) = \lim_{m\to\infty} (-)^m\, c_{2m+1}(x_j) = A\, q^{j^2/2}\, \theta_1(\frac{\pi j}{2}, \lambda) \tag{56}$$

where $\theta_i(z,\lambda)$ are the Jacobi-, or elliptic θ-functions defined as

$$\theta_1(z,\lambda) \;:=\; 2\lambda^{1/4}\sum_{n=0}^{\infty}(-)^n\lambda^{n(n+1)}\,\sin((2n+1)z) \tag{57}$$
$$\theta_2(z,\lambda) \;:=\; 2\lambda^{1/4}\sum_{n=0}^{\infty}\lambda^{n(n+1)}\,\cos((2n+1)z) \tag{58}$$

and where in Eqs.55,56 the constants λ and A are defined as

$$\lambda := e^{\frac{-\pi^2}{2\ln q}} \tag{59}$$

and

$$A^2 \;:=\; \frac{\pi}{2\,(q^{-2};q^{-2})^3_\infty\,\ln q} \;=\; \frac{2}{q^{\frac{1}{4}}\,\theta_2(0,q^{-1})\,\theta_2^2(0,\lambda)}\;. \tag{60}$$

4.2 Normalisation and scalar product of maximal localisation states

In order to evaluate the scalar product of two maximally localised states

$$\langle\psi_x^{ml}|\psi_{x'}^{ml}\rangle \;=\; \frac{1}{\sqrt{\mathcal{N}(x)\mathcal{N}(x')}}\sum_{n=0}^{\infty}q^{-3n}\,c_n(x)\,c_n(x') \tag{61}$$

the q-difference equation Eq.47 can be used to rewrite this expression as

$$\langle\psi_x^{ml}|\psi_{x'}^{ml}\rangle \;=\; \frac{q^{j+j'}f_{j-1,j'-1}\;+\;q^{j-j'}f_{j-1,j'+1}\;+\;q^{j'-j}f_{j+1,j'-1}\;+\;q^{-j-j'}f_{j+1,j'+1}}{(q^j+q^{-j})\,(q^{j'}+q^{-j'})\,\sqrt{\mathcal{N}(x)\mathcal{N}(x')}} \tag{62}$$

where we defined

$$f_{j,j'} \;:=\; \sum_{n=0}^{\infty}q^{-n}\,c_n(x)\,c_n(x') \tag{63}$$

and where we abbreviated $j := j(x)$ and $j' := j(x')$. We can compute $f_{j,j'}$ by applying the Christoffel-Darboux [21] theorem

$$\sum_{n=0}^{m}q^{-n}\,c_n(x)\,c_n(x') \;=\; \frac{2L\sqrt{[m+1]}}{q^{m+\frac{1}{2}}(q+q^{-1})}\;\frac{c_{m+1}(x)c_m(x')\;-\;c_m(x)c_{m+1}(x')}{x-x'} \tag{64}$$

which can be proved as follows. For $m>0$ (the case $m=0$ is trivial) we use the recursion relation (c.f. Eq. 25)

$$c_{m+1}(x) \;=\; \sqrt{\frac{q}{[m+1]}}\,\Big(\frac{q+q^{-1}}{2L}\,x\,c_m(x)\;-\;\sqrt{q[m]}\,c_{m-1}(x)\Big) \tag{65}$$

in order to replace $c_{m+1}(x)$ and $c_{m+1}(x')$ on the r.h.s. of Eq.64 which then takes the form

$$\text{r.h.s.} \ = \ q^{-m} \ c_m(x) \, c_m(x') \ - \ \frac{2L\sqrt{[m]}}{q^{m-\frac{1}{2}}(q+q^{-1})} \ \frac{c_{m-1}(x)c_m(x') \ - \ c_m(x)c_{m-1}(x')}{x-x'} \tag{66}$$

so that Eq.64 follows by induction.

Since the polynomials $c_m(x)$ have well defined limits as m goes to infinity, the Christoffel-Darboux theorem implies that the expression $f_{j,j'}$ is given by

$$f_{j,j'} \ = \ \sum_{n=0}^{\infty} q^{-n} \, c_n(x) \, c_n(x') \ = \ \frac{\sqrt{q}}{2\omega} \ \frac{c^-(x)c^+(x') - c^+(x)c^-(x')}{x-x'} \ . \tag{67}$$

Inserting Eqs.55,56 yields

$$f_{j,j'} \ = \ \frac{2\, A^2 \, q^{\frac{1}{2}(j^2+j'^2+1)}}{q^j - q^{-j} - q^{j'} + q^{-j'}} \ \theta_1\Big(\frac{\pi}{2}(j-j'),\lambda^2\Big) \ \theta_4\Big(\frac{\pi}{2}(j+j'),\lambda^2\Big) \tag{68}$$

with the definition of θ_4 being

$$\theta_4(z,\lambda^2) \ := \ 1 \ + \ 2\sum_{n=1}^{\infty} (-)^n \, \lambda^{2n^2} \ \cos(2nz) \ . \tag{69}$$

In the limit $x \to x'$ Eq.68 reduces to

$$f_{j,j} \ = \ \sum_{n=0}^{\infty} q^{-n} \, c_n^2(x) \ = \ \frac{q^{j^2+\frac{1}{4}} \, \theta_2(0,q^{-1})}{(q^j+q^{-j}) \, \theta_4(0,\lambda^2)} \, \theta_4(\pi j,\lambda^2) \ . \tag{70}$$

Inserting Eq.67 into Eq.62 we eventually obtain an exact expression for the scalar product of two quasi-position states:

$$\langle \psi_x^{ml} | \psi_{x'}^{ml} \rangle \ = \ \frac{2A^2 \, q^{\frac{1}{2}(j^2+j'^2+1)}(q^2-1)^2(1+q^{-2})}{\sqrt{\mathcal{N}(x)\mathcal{N}(x')} \, G^0_{j,j'} \, G^1_{j,j'} \, G^{-1}_{j,j'}} \, \theta_1(\frac{\pi}{2}(j'-j),\lambda^2) \, \theta_4(\frac{\pi}{2}(j+j'),\lambda^2) \tag{71}$$

where

$$G^s_{j,j'} \ = \ (q^{\frac{1}{2}(j-j')+s} - q^{-\frac{1}{2}(j-j')-s})(q^{\frac{1}{2}(j+j')+s} + q^{-\frac{1}{2}(j+j')-s}) \, . \tag{72}$$

Note that the poles in the denominator of Eq.71 are cancelled by the zeros of the θ_1-function. The limit $x \to x'$ yields the norm (Eq.23)

$$\mathcal{N}(x) \ = \ \frac{2\, q^{j^2} \, (q^2+1) \, \theta_4(\pi j,\lambda^2)}{A^2(q^j+q^{-j}) \, (q^{j+1}+q^{-j-1}) \, (q^{j-1}+q^{-j+1}) \, \theta_2^2(0,\lambda) \, \theta_4(0,\lambda^2)} \ . \tag{73}$$

4.3 Example: The quasi-position wave function of $|\psi_0^{ml}\rangle$.

As an example we draw the graph of the quasi-position wave function $\phi(x)$ for the state $|\phi\rangle$ that describes maximal localisation around $x = 0$ i.e. for $|\phi\rangle := |\psi_x^{ml}\rangle$, i.e. with

$$\phi(x) = \langle \psi_x^{ml} | \psi_0^{ml} \rangle \tag{74}$$

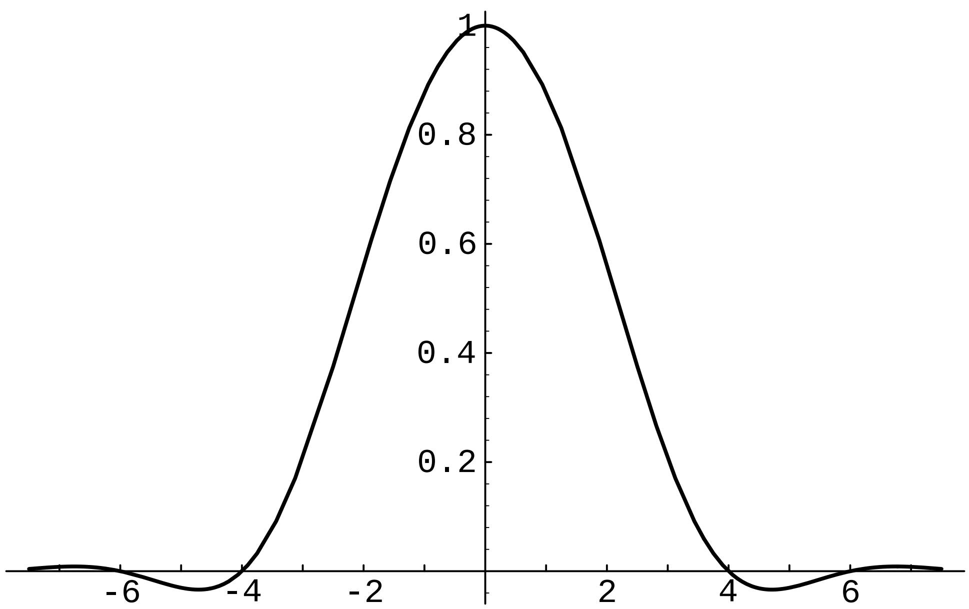

Fig. 1: Quasi-position wave function $\phi(x)$ for $|\phi\rangle := |\psi_0^{ml}\rangle$ for $q = 1.5$ drawn over $j(x)$

The analytic form of the wave function is given in Eq.71. For the width of the main peak note that the graph shows the overlap of pairs of localisation states, each with its finite position uncertainty.

We have thus generalised the treatment of [15] where the corresponding graph was calculated and drawn for the special case without a minimal momentum uncertainty ($\alpha = 0, \beta > 0$).

5 Approximations

For potential applications of the formalism the parameters α and β in Eq.2 can be assumed small in which case useful simplifications hold.

In the notation of Eq.3 this is the case when $q \to 1$. Then $\lambda \to 0$ and the θ-functions in Eqs.55,56 can be approximated by $\theta_1(\frac{\pi j}{2}, \lambda) \approx 2\lambda^{1/4} \sin \frac{\pi j}{2}$ and $\theta_2(\frac{\pi j}{2}, \lambda) \approx 2\lambda^{1/4} \cos \frac{\pi j}{2}$. This implies that

$$\begin{aligned} c^+(x_j) &\approx \tilde{c}^+(x_j) := B\, q^{j^2/2} \cos \frac{\pi j}{2} \\ c^-(x_j) &\approx \tilde{c}^-(x_j) := B\, q^{j^2/2} \sin \frac{\pi j}{2} \end{aligned} \tag{75}$$

where

$$B^4 = \frac{4 \ln q}{\pi \sqrt{q}} . \tag{76}$$

The relative error of this approximation is shown in the following table:

q	1.2	1.5	2	5
$\lvert 1 - \frac{\tilde{c}^\pm(x)}{c^\pm(x)} \rvert$	$< 5 \times 10^{-15}$	$< 1 \times 10^{-10}$	$< 2 \times 10^{-6}$	$< 7 \times 10^{-3}$

Using Eq.75 we can give approximations of the scalar product for q close to 1:

$$\langle \psi_x^{ml} | \psi_{x'}^{ml} \rangle \approx \frac{B^2\, q^{\frac{1}{2}(j^2+j'^2+1)} (q^2-1)^2 (1+q^{-2})}{\sqrt{\tilde{\mathcal{N}}(x) \tilde{\mathcal{N}}(x')}\, G^0_{j,j'}\, G^1_{j,j'}\, G^{-1}_{j,j'}} \sin \frac{\pi}{2}(j'-j) \tag{77}$$

where

$$\mathcal{N}(x) \approx \tilde{\mathcal{N}}(x) := \frac{2\, q^{j^2}\, (q^2+1)}{B^2 (q^j + q^{-j})\, (q^{j+1} + q^{-j-1})\, (q^{j-1} + q^{-j+1})} . \tag{78}$$

E.g. for $1 < q < 1.2$ the relative error of this approximation is less than 10^{-14}.

It is interesting to consider also the limiting case where

$$q \to 1, \qquad K(q) := \sqrt{\frac{q^2-1}{4\beta}}, \qquad L(q) = \hbar \frac{q^2+1}{2} \sqrt{\frac{\beta}{q^2-1}} . \tag{79}$$

In this limit the commutation relations Eq.3 turn into the relations Eq.2 with β finite but $\alpha = 0$, which is the special case considered in [15]. There is then only a minimal uncertainty in position and no minimal uncertainty in momentum. As can be shown easily, the limit $q \to 1$ of the scalar product Eqs.77-78 is given by

$$\lim_{q \to 1} \langle \psi_{x'}^{ml} | \psi_x^{ml} \rangle = \frac{\sin \frac{\pi}{2}(j'-j)}{\pi\, (\frac{j-j'}{2})\, (\frac{j-j'}{2}+1)\, (\frac{j-j'}{2}-1)} . \tag{80}$$

In the limit given by Eqs.79, x and j are related linearly through $x_j = x(j) = \hbar\sqrt{\beta}\, j$. We thus obtain the limiting expression for the scalar product:

$$\langle \psi^{ml}_{x'} | \psi^{ml}_{x} \rangle = \frac{1}{\pi} \left(\frac{x - x'}{2\hbar\sqrt{\beta}} - \left(\frac{x - x'}{2\hbar\sqrt{\beta}} \right)^3 \right)^{-1} \sin \left(\frac{x - x'}{2\hbar\sqrt{\beta}} \pi \right) \tag{81}$$

This result coincides with the expression found in [15], thus providing a nontrivial consistency check: We calculated the scalar product using q-analysis on a discrete q- Fock space representation. However, the calculation [15] of this scalar product in the special case $\alpha = 0$, which we here recover in the limit, had been performed with entirely different analytic methods in a continuous representation.

6 Self-adjoint extensions of x and p

In this section we continue formal considerations of [12] where it was proved that the operators $\mathbf{x}$ and $\mathbf{p}$ separately do have one-parameter families of self-adjoint extensions in H. To be precise, $\mathbf{x}$ on D is symmetric, while its adjoint $\mathbf{x}^*$ is closed but non-symmetric. $\mathbf{x}^{**}$ is closed and symmetric and has deficiency indices (1,1). There are families of diagonalisations of $\mathbf{x}$ in H, though of course not in D. The same holds for $\mathbf{p}$. The corresponding eigenvectors are unphysical states, separated from the physical domain by the minimal uncertainty gap, see Eq.15.

While in [12] the existence only of self-adjoint extensions had been proven, we can now explicitly solve the eigenvalue problem

$$\mathbf{x}.|v_x\rangle = x|v_x\rangle \tag{82}$$

For the solution we make the ansatz

$$|v_x\rangle \;=\; N^{-1}(x) \sum_{n=0}^{\infty} q^{-n/2} d_n(x) |n\rangle \tag{83}$$

yielding the recurrence relation

$$\frac{x}{L} d_n(x) \;=\; \sqrt{[n+1]q^{-1}}\, d_{n+1}(x) \;+\; \sqrt{q[n]}\, d_{n-1}(x) \tag{84}$$

with $d_{-1} = 0$ and $d_0 = 1$. In fact, Eq.84 can be transformed into the recurrence relation Eq.25, i.e. the d_n can be transformed into our previously considered coefficients c_n:

$$d_n(x) \;=\; c_n(2x(q + q^{-1})^{-1}) \;. \tag{85}$$

In the expansion of $|v_x\rangle$, the factor $q^{-n/2}$ is different from the corresponding factor $q^{-3n/2}$ in the expansion of the $|\psi^{ml}_x\rangle$, implying that the scalar product and normalisation constant of the formal eigenvectors are different from those of the maximal

localisation states which we had calculated earlier:

$$N(x) \;== \frac{q^{j^2+1/4}\,\theta_4(\pi j,\lambda^2)\,\theta_2(0,q^{-1})}{(q^j+q^{-j})\,\theta_4(0,\lambda^2)} \tag{86}$$

$$\langle v_x|v_{x'}\rangle = \frac{A^2 q^{1/2(j^2+j'^2+1)}\theta_1(\frac{\pi}{2}(j-j')\lambda^2)\,\theta_4(\frac{\pi}{2}(j+j'),\lambda^2)}{(x-x')\,\bar{\omega}\,\sqrt{N(x)N(x')}} \tag{87}$$

where now

$$x(j) := \frac{q^j - q^{-j}}{2\bar{\omega}} \qquad \text{with} \qquad \bar{\omega} = \frac{\sqrt{q^2-1}}{2L} \tag{88}$$

and where we abbreviated again $j' := j(x')$.
We draw the graph of the scalar product over j for $j' = 0$:

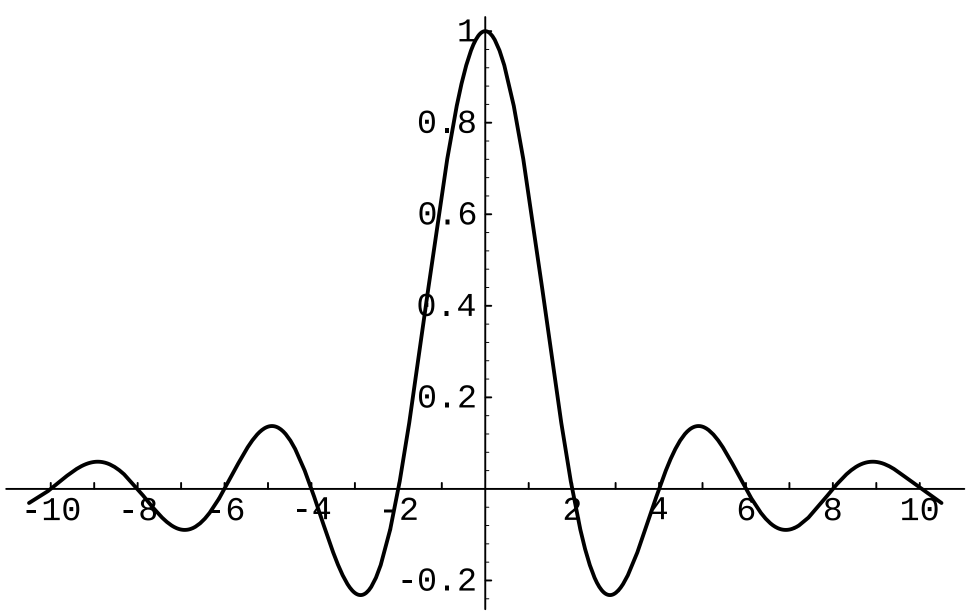

Fig. 2: Scalar product $\langle v_x|v_0\rangle$ of formal eigenvectors drawn over $j(x)$

From the zero's of θ_1 we read off that the $|v_x\rangle$ are mutually othogonal for $j - j' \in 2\mathbb{N}$. Using j' as a parameter in the range $j' \in [0, 2[$ we identify for each value of j' a diagonalisation of $\mathbf{x}$. Thus, j' labels the self-adjoint extensions with the corresponding eigenvalues $(x_n)_{n\in\mathbb{N}}$ being

$$x_n = \frac{q^{2n+j'} - q^{-2n-j'}}{2\sqrt{q^2-1}} L = \frac{\sinh((2n+j')\ln q)}{\sqrt{q^2-1}} L \qquad (n \in \mathbb{N}) \tag{89}$$

Compare also with the graph of the scalar product which had been calculated only numerically in [12]. Having found the analytic form of the scalar product in terms of θ functions we were able to determine the one parameter familiy of diagonalisations of $\mathbf{x}$, of which we had so far only known its existence. As is not difficult to see we recover for vanishing minimal uncertainty in momentum, i.e. for $\alpha \to 0$, the linear spectrum found in [15] for that special case.

Analogously to above we obtain the eigenvalues of $\mathbf{p}$ in its self-adjoint extensions ($j'' \in [0, 2[$):

$$p_n = \frac{q^{2n+j''} - q^{-2n-j''}}{2\sqrt{q^2-1}} K = \frac{\sinh((2n+j'')\ln q)}{\sqrt{q^2-1}} K \qquad (n \in \mathbb{N}) \tag{90}$$

We stress that the parameters j', j'' of Eqs.89,90 label *different* extensions of the domain D of $\mathbf{x}$ and $\mathbf{p}$. Recall that the uncertainty relation implies that the formal $\mathbf{x}$- or $\mathbf{p}$- eigenvectors which we here calculated do not lie in any *common* extension of the domain D of $\mathbf{x}$ and $\mathbf{p}$. They are not physical states and are separated from the physical domain by the uncertainty gap, see Eq.15.

However, these families of diagonalisations of $\mathbf{x}$ or $\mathbf{p}$ in H- can still be of use, e.g. for the calculation of inverses of $\mathbf{x}$ and $\mathbf{p}$, which would have been difficult to invert as nondiagonal operators in the Fock basis $\mathbf{x} = L(a + a^\dagger)$ and $\mathbf{p} = iK(a - a^\dagger)$.

7 Outlook

In quantum field theory, interaction terms which on ordinary geometry would be ultraviolet regular but nonlocal, can in fact be regular and strictly local on a geometry with a minimal position uncertainty. The reason is that an interaction is to be considered strictly local if no nonlocality could be observed. Intuitivly this is the case if a small apparent nonlocality of the interaction term is unobservable due to a comparatively larger minimal uncertainty in the underlying space. We already mentioned that, as has been shown in [13], quantum field theories can be naturally regularised when working on a generalised geometry with intrinsic minimal uncertainties. Generally, in order to explicitly compare the size of nonlocality of an arbitrary interaction term with the size of the intrinsic uncertainty of the generalised geometry it is crucial to have available the states of maximal localisation on this geometry. Similarly, maximal localisation in a momentum space with minimal uncertainty Δp_0 is of interest in the context of infrared regularisation.

So far, we have studied the properties of the maximal localisation states in one dimension only. The generalised Fourier transformations which map between the (quasi-) position and the (quasi-) momentum representations have only been studied in the special case $\alpha = 0$. For the general case techniques should be useful which have been developed for the Fourier theory [22] on quantum planes [23]. Also, the unitary equivalence of *all* Hilbert space representations, in the sense in which it holds for the ordinary commutation relations, has not yet been proven. Most interesting further physical insight into the nature of these generalised geometries can be expected from studies on maximal localisation in n dimensions where $[\mathbf{x}_i, \mathbf{x}_j] \neq 0$ and $[\mathbf{p}_i, \mathbf{p}_j] \neq 0$ lead also to $\Delta x_i \Delta x_j \neq 0$ and $\Delta p_i \Delta p_j \neq 0$. Work in this direction is in progress.

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8 Appendix A

In the following we outline a proof for the equivalence of the formula of Rodriguez Eq.43 and the recurrence relation Eq.25. We use the notation $Af(x_j) := \frac{1}{2}\Big(f(x_{j-1}) + f(x_{j+1})\Big)$ which allows the differentiation of products to be written in the form:

$$D(fg) \;=\; (Df)(Ag) + (Af)(Dg)\,. \tag{91}$$

We first prove the identity

$$D^{n+1} q^{-j^2} \;=\; -\Big(\frac{q+q^{-1}}{2L}\Big)^2 \left(q^{-2n+1}[n]\, D^{n-1} + x\, q^{-n} D^n\right) q^{-j^2}\,. \tag{92}$$

The case $n = 0$ can be verified easily by hand. Induction from n to $n+1$ implies that

$$D^{n+1} q^{-j^2} \;=\; D(D^n q^{-j^2}) \;=\; -D\left(\frac{q+q^{-1}}{2L}\right)^2 \left(q^{-2n+3}[n-1]\, D^{n-2} + x\, q^{-n+1} D^{n-1}\right) q^{-j^2} \tag{93}$$

Comparing the r.h.s. of Eqs.92 and 93 and using Eq.91, one is led to the condition

$$\left(A\, D^{n+1} + \frac{1}{2}(q-q^{-1})\, x\, D^n - q^{-n} D^{n-1}\right) q^{-j^2} \;=\; 0 \tag{94}$$

which can be proved by a further induction where the identities

$$Dx = 1\,, \qquad Ax = \frac{1}{2}(q+q^{-1})x\,, \qquad DA - \frac{1}{2}(q+q^{-1})AD = \frac{1}{4}(q-q^{-1})^2 x\, D^2$$

turn out to be very useful. Once Eq.92 is proved, one obtains the recursion relation Eq.25 by inserting the formula of Rodriguez which completes the proof.

9 Appendix B

We prove the limits $c^{\pm}(x)$ i.e. we prove Eqs.55,60:

$$c^+(x_j) \;=\; \lim_{m\to\infty} (-)^m\, c_{2m}(x_j) \;=\; \sqrt{\frac{\pi}{2\,(q^{-2};q^{-2})_\infty^3\,\ln q}}\, q^{j^2/2}\,\theta_2(\frac{\pi j}{2},\lambda) \tag{95}$$

Let us first rewrite expression Eq.38 by

$$c_{2m}(x_j) \;=\; \frac{1}{q^{2m^2}\sqrt{(q^{-2};q^{-2})_{2m}}} \sum_{k=-m}^{m} \binom{2m}{m+k}_{q^2} (-)^{m+k}\, q^{2kj}\,. \tag{96}$$

Choosing an integer $0 < r < m$ we split up this sum into two parts

$$(-)^m\, c_{2m}(x_j) = S^{(1)}_{m,r} + S^{(2)}_{m,r} \tag{97}$$

where

$$S^{(1)}_{m,r} \;=\; \frac{1}{q^{2m^2}\sqrt{(q^{-2};q^{-2})_{2m}}} \sum_{k=-r}^{r} \binom{2m}{m+k}_{q^2} (-)^{k}\, q^{2kj} \tag{98}$$

$$S^{(2)}_{m,r} \;=\; \frac{1}{q^{2m^2}\sqrt{(q^{-2};q^{-2})_{2m}}} \sum_{k=r+1}^{m} \binom{2m}{m+k}_{q^2} (-)^{k}\, (q^{2kj}+q^{-2kj}) \tag{99}$$

Now let m go to infinity and keep r fixed. Since $q^2 > 1$ and Eq.28 we have the identity

$$\binom{2m}{m+k}_{q^2} \;=\; q^{2(m^2-k^2)} \binom{2m}{m+k}_{q^{-2}}\,. \tag{100}$$

Because of

$$\lim_{m\to\infty} \binom{2m}{m+k}_{q^{-2}} \;=\; \frac{1}{(q^{-2};q^{-2})_\infty} \tag{101}$$

the first part converges to

$$S^{(1)}_r \;=\; \lim_{m\to\infty} S^{(1)}_{m,r} \;=\; \frac{1}{(q^{-2};q^{-2})_\infty^{3/2}} \sum_{k=-r}^{r} (-)^k\, q^{2kj-2k^2}\,. \tag{102}$$

The second part $S^2_{m,r}$ can be estimated as follows. As can be seen from Eq.100 the inequality

$$\binom{2m}{m+k}_{q^{-2}} \;=\; \frac{\prod_{i=m+k+1}^{2m}(1-q^{-2i})}{\prod_{i=1}^{m-k}(1-q^{-2i})} \;\le\; \frac{1}{\prod_{i=1}^{2m}(1-q^{-2i})} \;=\; \frac{1}{(q^{-2};q^{-2})_{2m}} \tag{103}$$

implies that

$$\binom{2m}{m+k}_{q^2} \;\le\; \frac{q^{2(m^2-k^2)}}{(q^{-2};q^{-2})_\infty}\,. \tag{104}$$

Therefore

$$|S^{(2)}_{m,r}| \leq \frac{2}{q^{2m^2}\sqrt{(q^{-2};q^{-2})_{2m}}} \sum_{k=r+1}^{m} \binom{2m}{m+k}_{q^2} q^{2kj} \leq \frac{2}{(q^{-2};q^{-2})^{3/2}_{2m}} \sum_{k=r+1}^{m} q^{2(kj-k^2)} \tag{105}$$

so that

$$|S^{(2)}_{r}| \;=\; |\lim_{m\to\infty} S^{(2)}_{m,r}| \;\leq\; \frac{2\,q^{2rj-2r^2}}{(q^{-2};q^{-2})^{3/2}_{\infty}} \sum_{k=1}^{\infty} q^{2(kj-k^2)}\,. \tag{106}$$

Since the sum on the r.h.s. is finite this expression tends to zero as r goes to infinity. Thus we conclude that

$$c^{+}(x_j) \;=\; \lim_{r\to\infty} S^{(1)}_{r} \;=\; \frac{1}{(q^{-2};q^{-2})^{3/2}_{\infty}} \sum_{k=-\infty}^{\infty} (-)^k\, q^{2kj-2k^2}\,. \tag{107}$$

The sum on the r.h.s. of this expression is essentially a Jacobi θ_2-function. In order to see this notice that its definition Eq.58 can also be written as

$$\theta_2(z, e^{-\tau}) \;=\; \sqrt{\frac{\pi}{\tau}} \sum_{k=0}^{\infty} (-)^k\, e^{-\frac{1}{\tau}(z-\pi k)^2}\,. \tag{108}$$

Inserting $\tau = -\ln\lambda = \frac{\pi^2}{2\ln q}$ and $z = \frac{\pi j}{2}$ we can express the sum in Eq.107 by

$$\sum_{k=-\infty}^{\infty} (-)^k\, q^{-2k^2+2kj} \;=\; \sqrt{\frac{\pi}{2\ln q}}\, q^{j^2/2}\, \theta_2(z,\lambda) \tag{109}$$

which completes the proof for $c^{+}(x_j)$. The proof for $c^{-}(x_j)$ follows the same lines.

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